Group classification of heat conductivity equations with a nonlinear source

1999 J. Phys. A: Math. Gen. 32 7405

(http://iopscience.iop.org/0305-4470/32/42/312)

J. Phys. A: Math. Gen. **32** (1999) 7405–7418. Printed in the UK PII: S0305-4470(99)05155-0

Group classification of heat conductivity equations with a nonlinear source

R Z Zhdanov† and V I Lahno‡

† Institute of Mathematics, 3 Tereshchenkivska Street, 252004 Kyiv, Ukraine
‡ Pedagogical Institute, 2 Ostrogradskogo Street, 314000 Poltava, Ukraine

E-mail: renat@imath.kiev.ua and lahno@pdpi.poltava.ua

Received 14 June 1999

Abstract. We suggest a systematic procedure for classifying partial differential equations (PDEs) invariant with respect to low-dimensional Lie algebras. This procedure is a proper synthesis of the infinitesimal Lie method, the technique of equivalence transformations and the theory of classification of abstract low-dimensional Lie algebras. As an application, we consider the problem of classifying heat conductivity equations in one variable with nonlinear convection and source terms. We have derived a complete classification of nonlinear equations of this type admitting nontrivial symmetry. It is shown that there are 3, 7, 28 and 12 inequivalent classes of PDEs of the type considered that are invariant under the one-, two-, three- and four-dimensional Lie algebras, correspondingly. Furthermore, we prove that any PDE belonging to the class under study and admitting the symmetry group of a dimension higher than four is locally equivalent to a linear equation. This classification is compared with existing group classifications of nonlinear heat conductivity equations and one of the conclusions is that all of them can be obtained within the framework of our approach. Furthermore, a number of new invariant equations are constructed which have rich symmetry properties and, therefore, may be used for mathematical modelling of, say, nonlinear heat transfer processes.

1. Introduction

Traditionally, group-theoretical (symmetry) analysis of differential equations has consisted of two interrelated problems. The first problem consists of finding the maximal Lie transformation (symmetry) group admitted by a given equation. The second problem is one of classifying differential equations that admit a prescribed symmetry group G. The principal tool for handling both problems is the classical infinitesimal routine developed by S Lie (see, e.g., [1–3]). It reduces the problem to finding the corresponding Lie symmetry algebra of infinitesimal operators whose coefficients are found as solutions of some over-determined system of linear partial differential equations (PDEs).

Solving a classification problem for some group G provides us with an exhaustive description of differential equations that are invariant with respect to this group and, consequently, allows analysis by means of the powerful Lie group technique. This is not just a matter of curiosity but the fundamental result that is used intensively in applications. An experimentalist, who believes that nature is governed by symmetry laws, is provided with a criterion (symmetry selection principle) for choosing a proper nonlinear model describing a real process under investigation. Normally, a researcher has some freedom in choosing the nonlinearities of the model and it would only be natural to take those nonlinearities that provide

0305-4470/99/427405+14$30.00

the highest symmetry for the model. The classical example is the Lorentz–Poincaré–Einstein relativity principle, which is to be respected by a physically meaningful model of relativistic field theory. From the point of view of the group theory the above principle is a requirement for a model under study to be invariant under the Poincaré group (for more details, see, e.g., [3,4]). Consequently, finding all possible Poincaré-invariant equations yields a complete account of all possible ways to model processes of relativistic field theory by PDEs.

In the overwhelming majority of papers devoted to solving classification problems, a representation of symmetry group G (symmetry algebra g) is fixed. Given this condition, the problem is solved by a straightforward application of the Lie algorithm. However, it becomes much more complicated if no specific representation of the symmetry algebra g is given. Then, utilizing the Lie algorithm directly, one arrives at the major difficulties coming from the necessity to find the maximal symmetry algebra and solve the classification problem, simultaneously. A principal idea enabling one to overcome the above difficulties was suggested by S Lie. Indeed, his method for obtaining all ordinary differential equations in one variable admitting non-trivial symmetry algebras [5, 6] instructs us what needs to be done in the case in question. First, we should construct all the possible inequivalent realizations of symmetry algebras within some class of Lie vector fields. If we succeed in doing this, then symmetry algebras will be specified, so that we can directly apply the Lie infinitesimal algorithm thus obtaining inequivalent classes of invariant equations. In this way, S Lie obtained his famous classification of realizations of all inequivalent complex Lie algebras on a plane [5,6]. Recently, Lie's classification has been used by Olver and Heredero [7] in order to obtain a classification of nonlinear wave equations in (1+1) dimensions that admit non-trivial spatial symmetries (i.e. symmetries not changing the temporal variable). What is more, Gonzalez-Lopez *et al* [8, 9] have classified quasi-exactly solvable models on a plane making use of their classification of real Lie algebras on a plane [5, 6].

A systematic implementation of these ideas for PDEs has been suggested by Ovsjannikov [1]. His approach is based on the concept of the equivalence group, which is the Lie transformation group acting in the properly extended space of independent variables, functions and their derivatives and preserving the class of PDEs under study. It is possible to modify the Lie algorithm in order to make it applicable for computing this group [1]. In the second step, the optimal system of subgroups of the equivalence group is constructed. The next step consists of utilizing the Lie algorithm for obtaining specific PDEs belonging to the class under study which are invariant with respect to the above-mentioned subgroups.

A further development of the Ovsjannikov approach has been undertaken by Akhatov *et al* [10, 11]. They obtained a number of classification results for nonlinear gas dynamics and diffusion equations. These ideas have also been utilized by Torrisi, Valenti and Tracina in order to perform preliminary group classification of some nonlinear diffusion and heat conductivity equations [12, 13]. Ibragimov and Torrisi have obtained a number of important results on the group classification of nonlinear detonation equations [14] and nonlinear hyperbolic-type equations [15]. Note that there is a number of papers (see, e.g., [16] and the references therein) devoted to a direct computation of equivalence groups of some PDEs. Being somewhat more involved, this approach has the merit of providing the possibility of finding *discrete* equivalence groups or even non-local ones.

The Ovsjannikov approach works smoothly provided an equivalence group is finite-dimensional. However, if the class of PDEs under study contains arbitrary functions of several arguments, then it could well be that it has an infinite-parameter equivalence group. The problem of subgroup classification of infinite-parameter Lie groups is completely open by now which makes a direct application of the Ovsjannikov approach problematic. Consequently, there is an evident need for the latter to be modified to become applicable to the case of infinite-parameter equivalence groups.

A possible way of modifying the Ovsjannikov approach is suggested by the manner in which physicists construct nonlinear generalizations of the linear wave equations. They take a specific representation of the Poincaré group realized on the solution set of the linear model and require that its nonlinear generalization should inherit this symmetry (for further details see, e.g. [3]). This approach makes the classification problem fairly easy to implement, since a representation of the symmetry algebra is fixed. A logical step forward is not to fix *a priori* a specific realization of the symmetry algebra but to fix the class of Lie vector fields within which this realization is searched for. It is this idea that enabled researchers to find principally new nonlinear realizations of the Euclid [4], Galilei [4, 17, 18], extended Galilei [17, 18], Schrödinger [17, 18], Poincaré [4] and extended Poincaré [19, 20] algebras. These results, in their turn, yield broad classes of new Galilei- and Poincaré-invariant nonlinear wave equations.

What we suggest in the present paper is a proper combination of the above-described approaches that enables a systematic treatment of a classification problem for the case of an infinite-parameter equivalence group admitted by the class of PDEs under study. We perform group classification for the class of parabolic-type equations describing nonlinear heat conductivity processes

$$u_t = u_{xx} + F(t, x, u, u_x) \tag{1.1}$$

where $u = u(t, x)$ is a smooth real-valued function, $u_t = \partial u/\partial t,\ u_x = \partial u/\partial x$ and so on, F is a sufficiently smooth real-valued function. As shown below, a direct application of the Ovsjannikov approach is not possible since the equivalence group admitted by the above equation is an infinite-parameter one. Due to this reason, a complete group classification has only been obtained for particular cases of (1.1) [21–25].

The paper has the following structure. In section 2 we introduce the general method and necessary definitions and notions. Section 3 is devoted to computing and analysing the equivalence group admitted by the class of PDEs (1.1). In section 4 we carry out the preliminary group classification of (1.1): namely, we give a complete description of locally inequivalent nonlinear PDEs of the form (1.1) that are invariant with respect to one-, two- and three-dimensional Lie algebras. In section 5 we present all inequivalent PDEs (1.1) admitting four-dimensional Lie algebras. Next, for each of the equations obtained we compute the maximal Lie symmetry algebra, thus obtaining the complete group classification of the corresponding models.

2. Description of the method

Our approach to group classification of PDEs is based on the following facts:

- A PDE having a nontrivial symmetry admits some finite- or infinite-dimensional Lie algebra of infinitesimal operators whose type is completely determined by the structure constants. Furthermore, if the symmetry algebra is infinite-dimensional, then, as a rule, it contains some finite-dimensional Lie algebra (for example, the centreless Virasoro algebra contains the algebra $sl(2, \mathbb{R})$).
- Abstract Lie algebras of up to five dimensions have already been classified [26–28].
- Equivalence transformations preserving the particular class of PDEs under study do not change the structure constants of the Lie algebra admitted.

Taking into account the above facts we formulate the following approach to group classification of nonlinear heat conductivity equations (1.1):

(1) First of all we find the most general form of infinitesimal operators admitted by PDEs (1.1). To this end we solve those determining equations that do not involve the function F. This yields a class $\mathcal{I}$ to which any symmetry of (1.1) should belong. Next, using an infinitesimal or direct approach we construct the equivalence group $G_{\mathcal{E}}$ of the class of PDEs (1.1). Evidently, the group $G_{\mathcal{E}}$ sets an equivalence relation on $\mathcal{I}$ (two elements of $G_{\mathcal{E}}$ are called equivalent if they are transformed into one another with a transformation from $G_{\mathcal{E}}$). We denote this relation as $\mathcal{E}$.

(2) In the second step, we find realizations of one-, two-, three-, four- and five-dimensional Lie algebras within the class $\mathcal{I}$ up to the equivalence relation $\mathcal{E}$. To this end we use the classification of low-dimensional abstract Lie algebras obtained by Mubarakzyanov [26, 27]

(3) Next, considering the obtained realizations of low-dimensional Lie algebras as symmetry algebras of the PDE (1.1) we classify all possible forms of functions F that provide invariance of the corresponding PDE with respect to this algebra. As a result, we get a complete classification of PDEs (1.1) admitting Lie symmetry algebras of up to five dimensions.

(4) In the final step, we apply the Lie infinitesimal algorithm for obtaining the maximal symmetry algebras admitted by those PDEs (1.1) that are invariant with respect to four- and five-dimensional Lie algebras. This is done straightforwardly, since the corresponding invariant PDEs (1.1) contains no arbitrary functions.

Note that the above approach does not allow for a complete group classification of PDEs (1.1), since there might exist realizations of higher symmetry algebras that do not contain four- or five-dimensional subalgebras. In fact, to get a full solution of classification problem one still has to be able to perform an exhaustive description of all inequivalent subalgebras of the Lie algebra of the infinite-parameter equivalence group $G_{\mathcal{E}}$. However, in the case under consideration our approach enables one to solve completely the group classification problem for (1.1), since there are essentially no nonlinear PDEs of the form (1.1) whose symmetry algebra has a dimension higher than four.

It is also clear, how to modify the above approach in order to classify PDEs admitting some prescribed symmetry algebra (say, the Galilei algebra). In the second step, one has to fix the corresponding structure constants and find all inequivalent realizations of the Galilei algebra within the class $\mathcal{I}$. Next, the maximal symmetry algebra is computed which yields the complete classification of Galilei-invariant PDEs of the form (1.1).

3. General analysis of symmetry properties of equation (1.1)

As the first step of the group classification of the PDE (1.1), we find the most general form of the infinitesimal operator of the Lie transformation group admitted. Furthermore, we construct the equivalence group of the class of PDEs (1.1).

Following the general Lie algorithm [1, 2] we are looking for an infinitesimal operator of the maximal symmetry group admitted by (1.1) in the form

$$Q = \tau\partial_t + \xi\partial_x + \eta\partial_u \tag{3.1}$$

where $\tau = \tau(t, x, u)$, $\xi = \xi(t, x, u)$, $\eta(t, x, u)$ are real-valued smooth functions defined in the space $X \otimes U$ of independent t, x and dependent $u = u(t, x)$ variables. The criterion for equation (1.1) to be invariant with respect to operator Q (3.1) reads as

$$(\varphi^t - \varphi^{xx} - \tau F_t - \xi F_x - \eta F_u - \varphi^x F_{u_x})|_{(1.1)} = 0. \tag{3.2}$$

Here

$$\varphi^t = D_t(\eta) - u_t D_t(\tau) - u_x D_t(\xi)$$
$$\varphi^x = D_x(\eta) - u_t D_x(\tau) - u_x D_x(\xi) \qquad (3.3)$$
$$\varphi^{xx} = D_x(\varphi^x) - u_{tx} D_x(\tau) - u_{xx} D_x(\xi)$$

where D_t, D_x are total differentiation operators defined in an appropriately prolonged space $X \otimes U$:

$$D_t = \partial_t + u_t\partial_u + u_{tt}\partial_{u_t} + u_{tx}\partial_{u_x} + \cdots$$
$$D_x = \partial_x + u_x\partial_u + u_{xx}\partial_{u_x} + u_{tx}\partial_{u_t} + \cdots. \qquad (3.4)$$

Splitting (3.2) in a usual way and solving equations that do not involve F, we get the forms of the coefficients τ, ξ of the operator Q

$$\tau = 2a(t) \qquad \xi = \dot{a}(t)x + b(t)$$

where $a(t), b(t)$ are arbitrary smooth functions and $\dot{a}(t) = \frac{da}{dt}$. Furthermore, the functions $a(t), b(t), \eta = f(t, x, u)$ and $F(t, x, u, u_x)$ have to satisfy the PDE

$$f_t - u_x(\ddot{a}x + \dot{b}) + (f_u - 2\dot{a})F = f_{xx} + 2u_x f_{xu} + u_x^2 f_{uu} + 2aF_t$$
$$+(\dot{a}x + b)F_x + fF_u + f_x F_{u_x} + u_x(f_u - \dot{a})F_{u_x}. \qquad (3.5)$$

Consequently, the maximal symmetry group admitted by equation (1.1) is generated by an infinitesimal operator of the form

$$Q = 2a(t)\partial_t + (\dot{a}(t)x + b(t))\partial_x + f(t, x, u)\partial_u \qquad (3.6)$$

functions a, b, f, F fulfilling relation (3.5).

Evidently, if we impose no restrictions on the choice of the function F, then the infinitesimal operator Q equals zero and, consequently, the symmetry group of the nonlinear heat conductivity equation (1.1) reduces to a trivial group of the identity transformations. Non-trivial symmetry groups appear, if we specify the source F in an appropriate way.

As we mentioned in the introduction, there are different ways to construct the equivalence group $G_{\mathcal{E}}$ for the class of PDEs (1.1). We use the direct method for finding the group $G_{\mathcal{E}}$.

Let

$$\tau = \alpha(t, x, u) \qquad \xi = \beta(t, x, u) \qquad v = \gamma(t, x, u) \qquad (3.7)$$

be an invertible change of variables that transforms the class of PDEs (1.1) into itself:

$$v_\tau = v_{\xi\xi} + G(\tau, \xi, v, v_\xi). \qquad (3.8)$$

Computing the derivative u_x yields

$$u_x = \frac{v_\tau\alpha_x + v_\xi\beta_x - \gamma_x}{\gamma_u - v_\tau\alpha_u - v_\xi\beta_u}.$$

On the other hand, in view of the arbitrariness of function F it follows from (3.8) that the relation of the form

$$u_x = g(\tau, \xi, v, v_\xi)$$

holds. Hence we conclude that in (3.7) $\alpha_x = \alpha_u = 0$, or $\alpha = \alpha(t)$, $\dot{\alpha} \equiv \frac{d\alpha}{dt} \neq 0$.

Computing the derivatives u_t, u_{xx} taking into account the relations $\alpha_x = \alpha_u = 0 \Leftrightarrow \alpha = \alpha(t), \dot{\alpha} \neq 0$ we get

$$u_t = v_\tau\dot{\alpha}(\gamma_u - v_\xi\beta_u)^{-1} + \theta_1(\tau, \xi, v, v_\xi)$$
$$u_{xx} = v_{\xi\xi}\{\beta_x^2(\gamma_u - v_\xi\beta_u)^{-1} + 2\beta_x\beta_u(v_\xi\beta_x - \gamma_x)(\gamma_u - v_\xi\beta_u)^{-2}$$
$$+\beta_u^2(v_\xi\beta_x - \gamma_x)^2(\gamma_u - v_\xi\beta_u)^{-3}\} + \theta_2(\tau, \xi, v, v_\xi)$$

with some function θ_2. Taking into consideration (3.8) yields the relation

$$\dot\alpha(\gamma_u - v_\xi\beta_u)^2 = \beta_x^2(\gamma_u - v_\xi\beta_u)^2 + 2\beta_x\beta_u(v_\xi\beta_x - \gamma_x)(\gamma_u - v_\xi\beta_u) + \beta_u^2(v_\xi\beta_x - \gamma_x)^2.$$

As α, γ, β do not depend on u_x, we can split the left-hand side of the above equation by v_ξ thus getting the system of determining equations for the functions α, β, γ

$$(\dot\alpha - \beta_x^2)\gamma_u^2 = \gamma_x\beta_u(\gamma_x\beta_u - 2\beta_x\gamma_u)$$
$$-2(\dot\alpha - \beta_x^2)\gamma_u\beta_u = 2\beta_x^2\gamma_u\beta_u$$
$$\dot\alpha\beta_u^2 = 0.$$

As $\dot\alpha \neq 0$, it follows from the last equation that $\beta_u = 0$. In view of this fact the system in question reduces to a single equation

$$(\dot\alpha - \beta_x^2)\gamma_u^2 = 0.$$

Since transformation of variables (3.7) is invertible, the relation $\gamma_u \neq 0$ holds. Hence we get $\dot\alpha = \beta_x^2$. Consequently, $\dot\alpha > 0$, $\beta = \pm\sqrt{\dot\alpha}x + \rho(t)$. Summing up, we conclude that the equivalence group $G_\mathcal{E}$ of the class of PDEs (1.1) reads as

$$\bar t = T(t) \qquad \bar x = \varepsilon\sqrt{\dot T(t)}x + X(t) \qquad \bar u = U(t, x, u) \tag{3.9}$$

where $\dot T(t) > 0$, $U_u \neq 0$, $\dot T = \frac{dT}{dt}$, $\varepsilon = \pm 1$.

Note that the infinitesimal method for finding the infinitesimal operator of the equivalence group yields the following class of operators (we skip the derivation of this formula):

$$E = \alpha(t)\partial_t + [\tfrac{1}{2}\dot\alpha(t)x + \rho(t)]\partial_x + \eta(t, x, u)\partial_u$$
$$+[\eta_t - \eta_{xx} + (\eta_u + \dot\alpha(t))F - u_x(\tfrac{1}{2}\ddot\alpha(t)x + \dot\rho(t)) - 2u_x\eta_{xu} - u_x^2\eta_{uu}]\partial_F \tag{3.10}$$

where $\alpha, \rho, \eta = \eta(t, x, u)$ are arbitrary smooth functions.

It is not difficult to convince oneself of the fact that transformations (3.9) can be obtained from the group transformations generated by operator (3.10) under the condition that the latter is complemented by the discrete transformation $x \to -x$. Consequently, both the direct and infinitesimal approaches give the same equivalence group for the class of nonlinear heat conductivity equations (1.1).

4. Preliminary group classification of equation (1.1)

In this section we classify equations of the form (1.1) that admit invariance algebras of up to three dimensions. We start from describing equations admitting one-dimensional Lie algebras, then proceed to investigation of the ones invariant with respect to two-dimensional algebras. Using these results we describe PDEs (1.1) which admit three-dimensional Lie algebras. An intermediate problem which is being solved, while classifying invariant equations of the form (1.1), is describing all possible realizations of one-, two- and three-dimensional Lie algebras by operators (3.6) within the equivalence relation (3.9). One more important remark is that PDEs that are equivalent to linear ones are excluded from further considerations. Note that we give the detailed calculations for the case of one-dimensional Lie algebras. For the higher-dimensional cases we present the final results, referring the reader interested in full details of the calculations to [29].

4.1. Nonlinear heat equations invariant under one-dimensional Lie algebras

All inequivalent realizations of one-dimensional Lie algebras having the basis elements of the form (3.6) are given by the lemma below.

Lemma 1. *There are diffeomorphisms (3.9) that reduce operator (3.6) to one of the following operators:*

$$Q = \pm\partial_t \tag{4.1}$$

$$Q = \partial_x \tag{4.2}$$

$$Q = \partial_u. \tag{4.3}$$

Proof. Let an operator Q have the form (3.6). Making the transformation (3.9) we have

$$Q \to \bar{Q} = 2a\dot{T}\partial_{\bar{t}} + \left[2a(\dot{X} + \tfrac{1}{2}x\ddot{T}(\dot{T})^{-\frac{1}{2}}) + \varepsilon(\dot{a}x + b)\sqrt{\dot{T}}\right]\partial_{\bar{x}}$$
$$+[2aU_t + (\dot{a}x + b)U_x + fU_u]\partial_{\bar{u}}.$$

In what follows, we have to differentiate between the cases $f = 0$ and $f \neq 0$, that is why they are considered separately.

Case 1. $f = 0$. Choosing $U = U(u)$ in (3.9) yields

$$\bar{Q} = 2a\dot{T}\partial_{\bar{t}} + [2a(\dot{X} + \tfrac{1}{2}x\ddot{T}(\dot{T})^{-\frac{1}{2}}) + \varepsilon(\dot{a}x + b)\sqrt{\dot{T}}]\partial_{\bar{x}}.$$

If $a = 0$, then $b \neq 0$ (since otherwise the operator Q is equal to zero). Therefore, choosing $T(t)$ in (3.9) as a solution of the equation $\dot{T} = |b(t)|^{-2}$ we arrive at the operator

$$\bar{Q} = \pm\partial_{\bar{x}}.$$

Within the space reflection $x \to -x$ we may choose Q' in the form $\bar{Q} = \partial_{\bar{x}}$.

Given the inequality $a \neq 0$, we put in (3.9) $\varepsilon = 1$. Choosing $T(t)$, $X(t)$ as solutions of the system of ordinary differential equations

$$\dot{T} - \frac{1}{2|a(t)|} = 0 \qquad 2a(t)\dot{X} + b(t)\sqrt{\dot{T}} = 0$$

we arrive at the operator

$$\bar{Q} = \pm\partial_{\bar{t}}.$$

Case 2. $f \neq 0$. Provided $a = b = 0$, we can choose U in (3.9) as a solution of the PDE $fU_u = 1$, thus getting the operator

$$\bar{Q} = \partial_{\bar{u}}.$$

If the inequality $|a| + |b| \neq 0$ holds, then choosing U in (3.9) as a solution of the PDE

$$2aU_t + (\dot{a}x + b)U_x + fU_u = 0 \qquad U_u \neq 0$$

we come to the above-considered case.

It is straightforward to check that the operators (4.1)–(4.3) cannot be transformed into one another with a change of variables (3.9). The lemma is proved. □

Consequently, there are three inequivalent one-dimensional Lie algebras

$$A_1^1 = \langle\epsilon\partial_t\rangle \qquad A_1^2 = \langle\partial_x\rangle \qquad A_1^3 = \langle\partial_u\rangle \qquad \epsilon = \pm 1.$$

A simple calculation shows that the corresponding invariant equations from the class (1.1) have the form

$$A_1^1 : u_t = u_{xx} + F(x, u, u_x) \tag{4.4}$$

$$A_1^2 : u_t = u_{xx} + F(t, u, u_x) \tag{4.5}$$

$$A_1^3 : u_t = u_{xx} + F(t, x, u_x). \tag{4.6}$$

4.2. Nonlinear heat equations invariant under two-dimensional Lie algebras

As is well known, there are two different abstract two-dimensional Lie algebras: namely, the commutative Lie algebra $A_{2.1} = \langle Q_1, Q_2\rangle$, $[Q_1, Q_2] = 0$ and the solvable one $A_{2.2} = \langle Q_1, Q_2\rangle, [Q_1, Q_2] = Q_2$.

Theorem 1. *The list of two-dimensional Lie algebras having the basis operators (3.6) and defined within the equivalence relation (3.9) is exhausted by the following algebras:*

$$A^1_{2.1} = \langle \partial_t, \partial_x\rangle \qquad A^2_{2.1} = \langle \partial_t, \partial_u\rangle$$
$$A^3_{2.1} = \langle \partial_x, \alpha(t)\partial_x + \partial_u\rangle \qquad A^4_{2.1} = \langle \partial_u, g(t,x)\partial_u\rangle \qquad g \neq const$$
$$A^5_{2.1} = \langle \partial_x, \alpha(t)\partial_x\rangle \qquad \dot\alpha \equiv \frac{d\alpha}{dt} \neq 0$$
$$A^1_{2.2} = \langle -t\partial_t - \tfrac{1}{2}x\partial_x, \partial_t\rangle \qquad A^2_{2.2} = \langle -2t\partial_t - x\partial_x, \partial_x\rangle$$
$$A^3_{2.2} = \langle -u\partial_u, \partial_u\rangle \qquad A^4_{2.2} = \langle \partial_x - u\partial_u, \partial_u\rangle$$
$$A^5_{2.2} = \langle \epsilon\partial_t - u\partial_u, \partial_u\rangle \qquad \epsilon = \pm 1.$$

Now we derive all inequivalent nonlinear heat conductivity equations (1.1), that admit two-dimensional Lie algebras as symmetry algebras.

For the realizations $A^1_{2.1}$ and $A^2_{2.1}$ the equations in question read as

$$A^1_{2.1} : u_t = u_{xx} + \tilde F(u, u_x) \qquad (4.7)$$
$$A^2_{2.1} : u_t = u_{xx} + \tilde F(x, u_x) \qquad (4.8)$$

correspondingly.

Given the realization $A^3_{2.1}$ we may use the result of (4.5) thus obtaining constraint (3.5) for the coefficient of the operator Q_2 in the form

$$-\dot\alpha u_x = F_u.$$

Hence it follows that

$$F = -\dot\alpha u u_x + \tilde F(t, u_x)$$

with an arbitrary smooth function $\tilde F$.

So the most general PDE (1.1) invariant with respect to the Lie algebra $A^3_{2.1}$ reads

$$A^3_{2.1} : u_t = u_{xx} - \dot\alpha u u_x + \tilde F(t, u_x). \qquad (4.9)$$

Treating the algebra $A^4_{2.1}$ in a similar way we represent constraint (3.5) as follows:

$$g_t = g_{xx} + g_x F_{u_x} \qquad g \neq \text{const}.$$

Given the relation $g_x = 0$, the function g is constant, i.e., $g = \text{const}$. This means that the PDE (1.1) becomes linear. To avoid this we should impose the restriction $g_x \neq 0$. Hence,

$$F = (g_t - g_{xx})g_x^{-1}u_x + \tilde F(t, x) \qquad g_x \neq 0.$$

Summing up, we conclude that the class of PDEs of the form (1.1) invariant with respect to the algebra $A^4_{2.1}$ reads as

$$A^4_{2.1} : u_t = u_{xx} + (g_t - g_{xx})g_x^{-1}u_x + \tilde F(t, x) \qquad g_x \neq 0. \qquad (4.10)$$

We now turn to the algebra $A^5_{2.1}$. Inserting the coefficients of the operator Q_2 into (3.5) yields

$$\dot\alpha u_x = 0$$

Table 1. Equations (1.1) admitting the algebras $A_{3.1}$, $A_{3.2}$.

Algebra	Function F
$A^1_{3.1}$	$G(u_x), \quad G_{u_x} \neq \lambda, \quad \lambda \in \mathbb{R}$
$A^1_{3.2}$	$u_x^2 G(\omega), \quad \omega = x u_x, \quad G \neq \lambda\omega^{-2}, \quad \lambda \in \mathbb{R}$
$A^2_{3.2}$	$t^{-1}G(\omega), \quad \omega = t u_x^2, \quad G \neq \lambda\sqrt{\omega}, \quad \lambda \in \mathbb{R}$
$A^3_{3.2}$	$-\frac{1}{2}t^{-1}u\sqrt{\lvert\omega\rvert} + t^{-1}G(\omega), \quad \omega = t u_x^2$
$A^4_{3.2}$	$-\dot{\alpha}\alpha^{-1}u_x \ln\lvert\omega\rvert + u_x G(t), \quad \dot{\alpha} \neq 0, \quad \omega = \mathrm{e}^x u_x$
$A^5_{3.2}$	$u_x G(\omega), \quad \omega = \mathrm{e}^x u_x, \quad G \neq \lambda\omega^{-1}, \quad \lambda \in \mathbb{R}$
$A^6_{3.2}$	$u_x G(\omega), \quad \omega = \mathrm{e}^{\epsilon t} u_x, \quad G \neq \lambda\omega^{-1}, \quad \lambda \in \mathbb{R}, \quad \epsilon = \pm 1$
$A^7_{3.2}$	$u_x G(\omega), \quad \omega = (u_x)^\lambda \mathrm{e}^{\epsilon(\lambda t - x)}, \quad \lambda > 0, \quad G \neq \text{const}, \quad \epsilon = \pm 1$

whence $\dot{\alpha} = 0$. This contradicts the assumption $\dot{\alpha} \neq 0$. Consequently, there are no equations of the form (1.1) admitting $A^5_{2.1}$ as a symmetry algebra.

Treating the algebras $A^i_{2.2}$ $(i = 1, \ldots, 5)$ in a similar way we get the following invariant equations:

$$A^1_{2.2} : u_t = u_{xx} + u_x^2 \tilde{F}(u, x u_x) \tag{4.11}$$

$$A^2_{2.2} : u_t = u_{xx} + t^{-1} \tilde{F}(u, t u_x^2) \tag{4.12}$$

$$A^3_{2.2} : u_t = u_{xx} + u_x \tilde{F}(t, x) \tag{4.13}$$

$$A^4_{2.2} : u_t = u_{xx} + u_x \tilde{F}(t, \mathrm{e}^x u_x) \tag{4.14}$$

$$A^5_{2.2} : u_t = u_{xx} + u_x \tilde{F}(x, \mathrm{e}^{\epsilon t} u_x) \qquad \epsilon = \pm 1 \tag{4.15}$$

where $\tilde{F}$ is an arbitrary smooth function.

Note that equations (4.10) and (4.13) are linear and, therefore, are excluded from further considerations.

4.3. Nonlinear heat equations invariant under three-dimensional Lie algebras

We split the set of abstract three-dimensional Lie algebras into two classes. The first class contains those algebras which are direct sums of lower-dimension ones. The remaining algebras are included into the second class.

4.3.1. Equation (1.1) invariant with respect to decomposable algebras. The first Lie algebra class contains two non-isomorphic algebras: namely, $A_{3.1}, A_{3.2}$. What is more, $A_{3.1} = \langle Q_1, Q_2, Q_3 \rangle$, $[Q_i, Q_j] = 0$ $(i, j = 1, 2, 3)$, i.e. $A_{3.1} = A_1 \oplus A_1 \oplus A_1 = 3A_1$ and $A_{3.2} = \langle Q_1, Q_2, Q_3 \rangle$, where $[Q_1, Q_2] = Q_2$, $[Q_1, Q_3] = [Q_2, Q_3] = 0$, i.e. $A_{3.2} = A_{2.2} \oplus A_1$.

We summarize the classification results of nonlinear heat conductivity equations (1.1) invariant under the three-dimensional Lie algebras belonging to the first class in table 1, where we use the following notations:

$$A^1_{3.1} = \langle \partial_t, \partial_x, \partial_u \rangle$$

$$A^1_{3.2} = \langle -t\partial_t - \tfrac{1}{2}x\partial_x, \partial_t, \partial_u \rangle$$

$$A^2_{3.2} = \langle -2t\partial_t - x\partial_x, \partial_x, \partial_u \rangle$$

$$A^3_{3.2} = \left\langle -2t\partial_t - x\partial_x, \partial_x, \sqrt{\lvert t\rvert}\partial_x + \partial_u \right\rangle$$

$$A^4_{3.2} = \langle \partial_x - u\partial_u, \partial_u, \alpha(t)\partial_x \rangle \qquad \dot{\alpha} \neq 0$$

$$A^5_{3.2} = \langle \partial_x - u\partial_u, \partial_u, \partial_t \rangle$$
$$A^6_{3.2} = \langle \epsilon\partial_t - u\partial_u, \partial_u, \partial_x \rangle$$
$$A^7_{3.2} = \langle \epsilon\partial_t - u\partial_u, \partial_u, \partial_t + \lambda\partial_x \rangle \qquad \lambda > 0$$

and what is more, $\epsilon = \pm 1$.

4.3.2. Equations (1.1) invariant with respect to non-decomposable algebras. Here we consider those three-dimensional real Lie algebras $A_3 = \langle Q_1, Q_2, Q_3 \rangle$ that cannot be decomposed into a direct sum of lower-dimensional Lie algebras. The list of these algebras is exhausted by the two semi-simple Lie algebras

$$A_{3.3} : [Q_1, Q_3] = -2Q_2 \qquad [Q_1, Q_2] = Q_1 \qquad [Q_2, Q_3] = Q_3$$
$$A_{3.4} : [Q_1, Q_2] = Q_3 \qquad [Q_2, Q_3] = Q_1 \qquad [Q_3, Q_1] = Q_2$$

and the nilpotent Lie algebra

$$A_{3.5} : [Q_2, Q_3] = Q_1 \qquad [Q_1, Q_2] = [Q_1, Q_3] = 0$$

and six solvable Lie algebras (non-zero commutation relations are given only)

$$A_{3.6} : [Q_1, Q_3] = Q_1 \qquad [Q_2, Q_3] = Q_1 + Q_2$$
$$A_{3.7} : [Q_1, Q_3] = Q_1 \qquad [Q_2, Q_3] = Q_2$$
$$A_{3.8} : [Q_1, Q_3] = Q_1 \qquad [Q_2, Q_3] = -Q_2$$
$$A_{3.9} : [Q_1, Q_3] = Q_1 \qquad [Q_2, Q_3] = qQ_2 \qquad (0 < |q| < 1)$$
$$A_{3.10} : [Q_1, Q_3] = -Q_2 \qquad [Q_2, Q_3] = Q_1$$
$$A_{3.11} : [Q_1, Q_3] = qQ_1 - Q_2 \qquad [Q_2, Q_3] = Q_1 + qQ_2 \qquad q > 0.$$

The classification results of nonlinear heat conductivity equations (1.1) admitting the three-dimensional Lie algebras from the second class are summarized in table 2, where the following notations are used:

$$A^1_{3.3} = \langle \partial_t, t\partial_t + \tfrac{1}{2}x\partial_x, -t^2\partial_t - tx\partial_x + x^2\partial_u \rangle$$
$$A^1_{3.5} = \langle \partial_x, \partial_t, t\partial_x + \partial_u \rangle$$
$$A^2_{3.5} = \langle \partial_u, \partial_t, t\partial_u + \lambda\partial_x \rangle \qquad \lambda > 0$$
$$A^3_{3.5} = \langle \partial_u, \partial_x, x\partial_u + b(t)\partial_x \rangle \qquad \dot{b} \neq 0$$
$$A^4_{3.5} = \langle \partial_u, \partial_x, x\partial_u + \lambda\partial_t \rangle \qquad \lambda \neq 0$$
$$A^5_{3.5} = \langle \partial_u + 2\lambda t\partial_x, \partial_x, x\partial_u + 2\lambda t[t\partial_t + x\partial_x - u\partial_u] \rangle \qquad \lambda \neq 0$$
$$A^1_{3.6} = \langle \partial_u, \partial_t, t\partial_t + \tfrac{1}{2}x\partial_x + (u+t)\partial_u \rangle$$
$$A^2_{3.6} = \langle \partial_x, \partial_u - \tfrac{1}{2}\ln|t|\partial_x, 2t\partial_t + x\partial_x + u\partial_u \rangle$$
$$A^3_{3.6} = \langle \partial_u, \partial_x, 2t\partial_t + x\partial_x + (u+x)\partial_u \rangle$$
$$A^4_{3.6} = \langle \partial_u, \alpha\partial_x, \alpha^2(\dot{\alpha})^{-1}\partial_t + (1+\alpha)x\partial_x + [(1-\alpha)u + x]\partial_u \rangle \qquad \alpha = \alpha(t) \qquad \dot{\alpha} \neq 0$$
$$\text{and} \qquad \alpha^2\ddot{\alpha} + 2(\dot{\alpha})^2 = 0$$
$$A^1_{3.7} = \langle \partial_t, \partial_u, t\partial_t + \tfrac{1}{2}x\partial_x + u\partial_u \rangle$$
$$A^2_{3.7} = \langle \partial_x, \partial_u, 2t\partial_t + x\partial_x + u\partial_u \rangle$$
$$A^1_{3.8} = \langle \partial_t, \partial_u, t\partial_t + \tfrac{1}{2}x\partial_x - u\partial_u \rangle$$
$$A^2_{3.8} = \langle \partial_x, \partial_u + \lambda t\partial_x, 2t\partial_t + x\partial_x - u\partial_u \rangle \qquad \lambda \in \mathbb{R}$$
$$A^1_{3.9} = \langle \partial_t, \partial_x, t\partial_t + \tfrac{1}{2}x\partial_x \rangle$$

Table 2. Equations (1.1) admitting three-dimensional Lie algebras from the second class.

Algebra	Function F
$A^1_{3.3}$	$\frac{1}{4}u_x^2 - x^{-1}u_x + x^{-2}G(\omega), \quad \omega = 2u - xu_x$
$A^1_{3.5}$	$-uu_x + G(u_x)$
$A^2_{3.5}$	$\lambda^{-1}x + G(u_x), \quad \lambda > 0, \quad G_{u_xu_x} \neq 0$
$A^3_{3.5}$	$-\frac{1}{2}\dot{b}(t)u_x^2 + G(t), \quad \dot{b} \neq 0$
$A^4_{3.5}$	$G(\omega), \quad \omega = t - \lambda u_x, \quad \lambda \neq 0 \quad G_{\omega\omega} \neq 0$
$A^5_{3.5}$	$-2\lambda uu_x + t^{-3}G(\omega), \quad \omega = u_x t^2 - \frac{t}{2\lambda}, \quad \lambda \neq 0$
$A^1_{3.6}$	$2\ln\|u_x\|G(\omega), \quad \omega = x^{-1}u_x$
$A^2_{3.6}$	$\frac{1}{2}t^{-1}uu_x + \|t\|^{-\frac{1}{2}}G(u_x),$
$A^3_{3.6}$	$\|t\|^{-\frac{1}{2}}G(\omega), \quad \omega = t^{-1}u_x^2, \quad G \neq \text{const}, \sqrt{\omega}$
$A^4_{3.6}$	$-\dot{\alpha}uu_x + \alpha^{-6}\exp(2\alpha^{-1})G(\omega), \quad \omega = u_x\alpha^4 - \frac{2}{3}\alpha^3$
$A^1_{3.7}$	$G(\omega), \quad \omega = x^{-1}u_x, \quad G_{\omega\omega} \neq 0$
$A^2_{3.7}$	$\|t\|^{-\frac{1}{2}}G(u_x), \quad G_{u_xu_x} \neq 0$
$A^1_{3.8}$	$x^{-4}G(\omega), \quad \omega = x^3u_x, \quad G_{\omega\omega} \neq 0$
$A^2_{3.8}$	$-\lambda uu_x + \|t\|^{-\frac{3}{2}}G(\omega), \quad \omega = tu_x, \quad \lambda \in \mathbb{R}, \quad \lambda^2 + G_{\omega\omega} \neq 0$
$A^1_{3.9}$	$u_x^2G(u), \quad G_u \neq 0$
$A^2_{3.9}$	$G(\omega), \quad \omega = u^{-1}u_x^2, \quad G_\omega \neq 0$
$A^3_{3.9}$	$x^{2(q-1)}G(\omega), \quad \omega = x^{1-2q}u_x, \quad G_{\omega\omega} \neq 0$
$A^4_{3.9}$	$-\frac{1}{2}\lambda(1-q)\|t\|^{-\frac{1}{2}(1+q)}uu_x + \|t\|^{\frac{1}{2}(q-2)}G(\omega), \ \omega = \|t\|^{\frac{1}{2}(1-q)}u_x, \quad \lambda^2 + G^2_{\omega\omega} \neq 0$
$A^1_{3.10}$	$-\lambda uu_x + (t^2+\lambda^{-2})^{-\frac{3}{2}}G(\omega), \quad \omega = \lambda u_x(t^2+\lambda^{-2}) - t, \quad \lambda \neq 0$
$A^1_{3.11}$	$-\dot{\alpha}uu_x + (1+\alpha^2)^{-\frac{3}{2}}\exp(q\arctan\alpha)G(\omega), \quad \omega = u_x(1+\alpha^2) - \alpha$

$$A^2_{3.9} = \langle \partial_t, \partial_x, t\partial_t + \tfrac{1}{2}x\partial_x + u\partial_u \rangle$$
$$A^3_{3.9} = \langle \partial_t, \partial_u, t\partial_t + \tfrac{1}{2}x\partial_x + qu\partial_u \rangle \qquad q \neq 0, \pm 1$$
$$A^4_{3.9} = \langle \partial_x, \partial_u + \lambda|t|^{\frac{1}{2}(1-q)}\partial_x, 2t\partial_t + x\partial_x + qu\partial_u \rangle \qquad 0 < |q| < 1 \qquad \lambda \in \mathbb{R}$$
$$A^1_{3.10} = \langle \partial_x, \lambda t\partial_x + \partial_u, -\lambda(t^2+\lambda^{-2})\partial_t - \lambda tx\partial_x + (\lambda tu - x)\partial_u \rangle \qquad \lambda \neq 0$$
$$A^1_{3.11} = \langle \partial_x, \alpha\partial_x + \partial_u, -(\dot{\alpha})^{-1}(1+\alpha^2)\partial_t + (q-\alpha)x\partial_x + [(\alpha+q)u - x]\partial_u \rangle$$
$$q > 0 \qquad \alpha = \alpha(t) \qquad \dot{\alpha} \neq 0 \qquad \text{and} \qquad (1+\alpha^2)\ddot{\alpha} = 2q(\dot{\alpha})^2.$$

Note that the nonlinear ordinary differential equations

$$\alpha^2\ddot{\alpha} + 2(\dot{\alpha})^2 = 0 \tag{4.16}$$
$$(1+\alpha^2)\ddot{\alpha} = 2q(\dot{\alpha})^2 \tag{4.17}$$

can be solved by quadratures. However, their general solutions are defined implicitly and cannot be expressed via elementary functions.

The general solution of (4.16) reads as

$$\int^\alpha \exp(-2\xi^{-1})\,\mathrm{d}\xi = \lambda t + \lambda_1 \qquad \{\lambda, \lambda_1\} \subset \mathbb{R} \qquad \lambda \neq 0$$

and the general solution of (4.17) is given by the formula

$$\int^\alpha \exp(-2q\arctan\xi)\,\mathrm{d}\xi = \lambda t + \lambda_1 \qquad \{\lambda, \lambda_1\} \subset \mathbb{R} \qquad \lambda \neq 0.$$

Table 3. Nonlinear PDEs (1.1) admitting four-dimensional Lie algebras.

No	Equation	Maximal invariance algebra
1	$u_t = u_{xx} + \frac{\lambda\epsilon u_x}{4\sqrt{\lvert t\rvert}} \ln \lvert tu_x^2\rvert + \frac{\beta u_x}{\sqrt{\lvert t\rvert}}$,	A_4^1
	$\epsilon = 1$ for $t > 0$, $\epsilon = -1$ for $t < 0$, $\beta \in \mathbb{R}$, $\lambda \neq 0$	
2	$u_t = u_{xx} - \lambda u_x(x + \ln \lvert u_x\rvert)$, $\lambda \neq 0$	A_4^2
3	$u_t = u_{xx} + \lambda \exp(-u_x)$, $\lambda \neq 0$	A_4^3
4	$u_t = u_{xx} + 2 \ln \lvert u_x\rvert$	A_4^4
5	$u_t = u_{xx} - u_x \ln \lvert u_x\rvert + \lambda u_x$, $\lambda \in \mathbb{R}$	A_4^5
6	$u_t = u_{xx} + \lambda u_x^{\frac{2k-2}{2k-1}}$, $\lambda \neq 0$, $k \neq 0, \frac{1}{2}, 1$	A_4^6
7	$u_t = u_{xx} + \frac{1}{4t} u_x^2$	A_4^7
8	$u_t = u_{xx} - uu_x + \lambda \lvert u_x\rvert^{\frac{3}{2}}$	
	$\lambda \neq 0$	A_4^8
	$\lambda = 0$	A_5
9	$u_t = u_{xx} + \lambda^{-1}x + m\sqrt{\lvert u_x\rvert}$, $\lambda > 0$, $m \neq 0$	A_4^9
10	$u_t = u_{xx} - \frac{\lambda\epsilon}{4}(1-q)\lvert t\rvert^{-\frac{1}{2}(1+q)} u_x^2$	
	$\lambda \neq 0$, $\lvert q\rvert \neq 1$, $\epsilon = 1$, $t > 0$, $\epsilon = -1$, $t < 0$	A_4^{10}
11	$u_t = u_{xx} + m\sqrt{\lvert t - \lambda u_x\rvert}$, $\lambda \cdot m \neq 0$	A_4^{11}
12	$u_t = u_{xx} - \frac{1}{2}\dot{\alpha} u_x^2 + (\lambda - \alpha)(1 + \alpha^2)^{-1}$, $\lambda \in \mathbb{R}$	A_4^{12}

5. Complete group classification of equations (1.1) invariant under four-dimensional Lie algebras

In this section we carry out group classification of nonlinear heat conductivity equations (1.1) admitting four-dimensional Lie algebras. To this end, we use the known classification of abstract four-dimensional Lie algebras [26] and the above-obtained classification of three-dimensional Lie algebras which are symmetry algebras of nonlinear heat equations of the form (1.1). Furthermore, for each invariant equation we compute the maximal symmetry algebra in Lie's sense, thus completing the classification.

We present all the results on classification of inequivalent essentially nonlinear PDEs (1.1) that are invariant with respect to four-dimensional Lie algebras (decomposable and non-decomposable) in table 3, where we use the following notations:

$$A_4^1 = \left\langle -2t\partial_t - x\partial_x, \partial_x, -u\partial_u + \lambda\sqrt{|t|}\partial_x, \partial_u \right\rangle \qquad \lambda \neq 0$$

$$A_4^2 = \left\langle \partial_x - u\partial_u, \partial_u, \frac{1}{\lambda}\partial_t, \mathrm{e}^{\lambda t}\partial_x \right\rangle \qquad \lambda \neq 0$$

$$A_4^3 = \langle \partial_t, \partial_u, \partial_x, 2t\partial_t + x\partial_x + (u + x)\partial_u \rangle$$

$$A_4^4 = \langle \partial_x, \partial_u, \partial_t, t\partial_t + \tfrac{1}{2}x\partial_x + (u + t)\partial_u \rangle$$

$$A_4^5 = \langle \partial_u, \partial_x, \partial_t, t\partial_x + u\partial_u \rangle$$

$$A_4^6 = \langle \partial_t, \partial_x, \partial_u, t\partial_t + \tfrac{1}{2}x\partial_x + ku\partial_u \rangle \qquad k \neq 0, \tfrac{1}{2}, 1$$

$$A_4^7 = \langle \partial_u, \partial_x, x\partial_u - \tfrac{1}{2}\ln|t|\partial_x, 2t\partial_t + x\partial_x + 2u\partial_u \rangle$$

$$A_4^8 = \langle \partial_x, \partial_t, t\partial_x + \partial_u, t\partial_t + \tfrac{1}{2}x\partial_x - \tfrac{1}{2}u\partial_u \rangle$$

$$A_4^9 = \langle \partial_u, \partial_t, t\partial_u + \lambda\partial_x, t\partial_t + \tfrac{1}{2}x\partial_x + \tfrac{3}{2}u\partial_u \rangle \qquad \lambda > 0$$

$$A_4^{10} = \langle \partial_u, \partial_x, x\partial_u + \lambda|t|^{\frac{1}{2}(1-q)}\partial_x, 2t\partial_t + x\partial_x + (1 + q)u\partial_u \rangle \qquad |q| \neq 1 \qquad \lambda \neq 0$$

$A_4^{11} = \langle \partial_u, \partial_t, x\partial_u + \lambda\partial_t, 2t\partial_t + x\partial_x + 3u\partial_u \rangle, \ \lambda \neq 0$

$A_4^{12} = \langle \partial_u, \partial_x, x\partial_u + \alpha\partial_x, -(\dot{\alpha})^{-1}(1+\alpha^2)\partial_t + (q-\alpha)x\partial_x + [2qu - \frac{1}{2}x^2]\partial_u \rangle$

where $q > 0$ $\quad \alpha = \alpha(t)$ $\quad$ and $\dot{\alpha} \neq 0$ $\quad$ is a solution of (4.17)

$A_5 = \langle \partial_x, t\partial_x + \partial_u, \partial_t, -2t\partial_t - x\partial_x + u\partial_u, t^2\partial_t + tx\partial_x - (tu - x)\partial_u \rangle.$

Note that, except for case 8 with $\lambda = 0$ the four-dimensional Lie algebras given in table 3 are maximal symmetry algebras of the corresponding PDEs (for more details, see [29]). Case 8 with $\lambda = 0$ gives rise to the Burgers equation that is linearizable through the (non-local) Cole–Hopf substitution.

6. Concluding remarks

We have established that there are three classes of equations (1.1) invariant with respect to one-parameter groups (formulae (4.4)–(4.6)), seven classes of equations (1.1) invariant with respect to two-parameter groups (formulae (4.7)–(4.9), (4.11), (4.12), (4.14), (4.15)), 28 classes of equations (1.1) invariant with respect to three-parameter groups (tables 1 and 2) and 12 classes of equations (1.1) invariant with respect to four-parameter groups (table 3). Furthermore, we have proved that the four-dimensional Lie algebras given in table 3 are maximal symmetry algebras in Lie's sense admitted by the corresponding nonlinear heat equations with the only exception being the linearizable Burgers equation.

In [29] we have proved that there are essentially no nonlinear equations of the form (1.1) that admit invariance algebra of a dimension higher than four. To this end we utilize the Levi–Maltsev theorem, some facts from the general theory of simple, semi-simple and solvable Lie algebras and the above classification of inequivalent realizations of one-, two-, three- and four-dimensional Lie algebras on the set of solutions of PDE (1.1). Consequently, the group classification of invariant PDEs (1.1) obtained in the present paper is complete.

It is shown in [29] that the existing group classifications of subclasses of equations of the form (1.1), that are due to Oron and Rosenau [22, 23], Dorodnitsyn [21], Serov and Cherniha [25] and Gandarias [24], can be obtained from our classification as particular cases.

In the present paper we concentrate on studying essentially nonlinear heat conductivity equations since the linear case is well investigated. However, it is fairly simple to recover the corresponding results within the framework of our approach (see, e.g., [29]).

When classifying invariant equations (1.1) we utilize as equivalence transformations local transformations of dependent and independent variables. Using non-local transformations, on the one hand, may result in a reduction of a number of equivalence classes and, on the other hand, may yield so-called quasi-local symmetries (for more detail on quasi-local symmetries see, e.g., [11]). Consider, as an example, the following subclass of PDEs of the form (1.1):

$$u_t = u_{xx} + f_1(t)u + f_2(t, x, u_x) \tag{6.1}$$

with arbitrary smooth functions f_1, f_2. If we differentiate (1.1) with respect to x and make a change of the dependent variable

$$u_x(t, x) \to v(t, x) \tag{6.2}$$

then we get a subclass of quasi-linear PDEs of the form (1.1)

$$v_t = v_{xx} + f_1(t)v + f_{2x}(t, x, v) + f_{2v}(t, x, v)v_x. \tag{6.3}$$

Evidently, the above two classes of PDEs (6.1) and (6.2) are inequivalent in the sense of the definition given in section 3, since transformation (6.2) is not local.

The technique developed in the present paper can be efficiently applied to carry out group classification of arbitrary classes of PDEs in two independent variables, since their maximal symmetry algebras are, as a rule, low-dimensional and we can use the classification of abstract low dimensional Lie algebras.

These and the related problems are now under study and the results will be reported in our future publications.

References

[1] Ovsjannikov L V 1982 *Group Analysis of Differential Equations* (New York: Academic)
[2] Olver P J 1986 *Applications of Lie Groups to Differential Equations* (Berlin: Springer)
[3] Fushchych W I, Shtelen W M and Serov N I 1989 *Symmetry Analysis and Exact Solutions of Nonlinear Equations of Mathematical Physics* (Kiev: Naukova Dumka)
Fushchych W I, Shtelen W M and Serov N I 1993 *Symmetry Analysis and Exact Solutions of Nonlinear Equations of Mathematical Physics* (Dordrecht: Kluwer) (English transl.)
[4] Fushchych W I and Zhdanov R Z 1997 *Symmetries and Exact Solutions of Nonlinear Dirac Equations* (Kyiv: Naukova Ukraina)
[5] Lie S 1924 *Gesammelte Abhandlungen* vol 5 (Leipzig: Teubner) pp 767–73
[6] Lie S 1927 *Gesammelte Abhandlungen* vol 6 (Leipzig: Teubner) pp 1–94
[7] Olver P J and Heredero R H 1996 *J. Math. Phys.* **37** 6419–38
[8] González–López A, Kamran N and Olver P J 1991 *J. Phys. A: Math. Gen.* **24** 3995–4008
[9] González–López A, Kamran N and Olver P J 1994 *Commun. Math. Phys.* **159** 503–37
[10] Akhatov I S, Gazizov R K and Ibragimov N K 1987 *Proc. Acad. Sci. USSR* **293** 1033–5
[11] Akhatov I S, Gazizov R K and Ibragimov N K 1989 *Sovremennye Problemy Matematiki. Novejshie Dostizheniya* vol 34 (Moscow: Nauka) pp 3–83
[12] Torrisi M, Tracina R and Valenti A 1996 *J. Math. Phys.* **37** 4758–67
[13] Torrisi M and Tracina R 1998 *Int. J. Nonlinear Mech.* **33** 473–87
[14] Ibragimov N H, Torrisi M and Valenti A 1991 *J. Math. Phys.* **32** 2988–95
[15] Ibragimov N K and Torrisi M 1992 *J. Math. Phys.* **33** 3931–7
[16] Kingston J G and Sophocleous C 1998 *J. Phys. A: Math. Gen.* **31** 1595–619
[17] Rideau G and Winternitz P 1993 *J. Math. Phys.* **34** 558–70
[18] Zhdanov R Z and Fushchych W I 1997 *J. Non. Math. Phys.* **4** 426–35
[19] Rideau G and Winternitz P 1990 *J. Math. Phys.* **31** 1095–105
[20] Fushchych W I and Lahno V I 1996 *Proc. Acad. Sci. Ukraine* no 11, 60–5
[21] Dorodnitsyn V A 1982 *Zhurn. Vych. Matemat. Matem. Fiziki* **22** 1393–400
[22] Oron A and Rosenau P 1986 *Phys. Lett.* A **118** 172–6
[23] Edwards M P 1994 *Phys. Lett.* A **190** 149–54
[24] Gandarias M L 1996 *J. Phys. A: Math. Gen.* **29** 607–33
[25] Serov M I and Cherniha R M 1997 *Ukrain. Math. J.* **49** 1262–70
[26] Mubarakzyanov G M 1963 *Izv. Vyssh. Uchebn. Zaved. Mat.* **32** 114–23
[27] Mubarakzyanov G M 1963 *Izv. Vyssh. Uchebn. Zaved. Mat.* **34** 99–106
[28] Turkowski P 1988 *J. Math. Phys.* **29** 2139–44
[29] Zhdanov R Z and Lahno V I 1999 *Preprint* math-ph/9906003